2008 Mathematics

Advanced Higher

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question 4, M1 means a method mark for correct form of partial fraction. E is shorthand for error. For example, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Mathematics 2008

1. Let the common difference be d. General term is $a + (n - 1)d$. **1**

So $2 + 19d = 97 \Rightarrow d = 5$. **1**

Sum of an arithmetic series is $\frac{n}{2}\left[2a + (n - 1)d\right]$. **1**

Required sum is $\frac{50}{2}\left\{4 + 49 \times 5\right\} = 6225$. **1**

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2. (a)

$$f(x) = \cos^{-1}(3x)$$

$$f'(x) = \frac{-1}{\sqrt{1 - (3x)^2}} \times 3 \qquad \textbf{1,1}$$

$$= \frac{-3}{\sqrt{1 - 9x^2}}$$

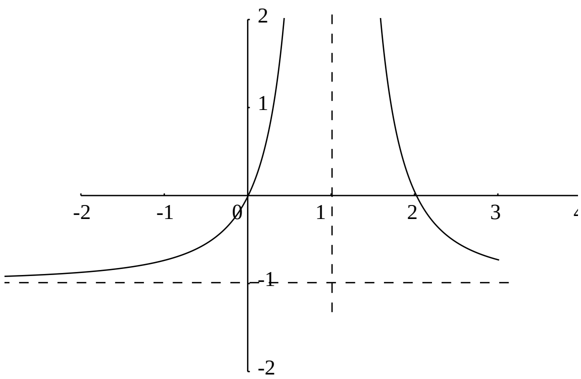
(b)

$$\frac{dx}{d\theta} = 2\sec\theta\tan\theta, \qquad \frac{dy}{d\theta} = 3\cos\theta \qquad \textbf{1,1}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta}{2\sec\theta\tan\theta} \qquad \textbf{1}$$

$$= \frac{3\cos^3\theta}{2\sin\theta}$$

3.



1 for inverting

1 for translation

1 for showing asymptotes

Asymptotes are $y = -1$ and $x = 1$. **1**

4.

$$\frac{12x^2 + 20}{x(x^2 + 5)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 5} \qquad \textbf{1M}$$

$$12x^2 + 20 = A(x^2 + 5) + x(Bx + C)$$

$$= (A + B)x^2 + Cx + 5A$$

$$\therefore \quad 5A = 20 \Rightarrow A = 4 \Rightarrow B = 8$$

$$C = 0 \qquad \textbf{2E1}$$

Hence
$$\frac{12x^2 + 20}{x(x^2 + 5)} = \frac{4}{x} + \frac{8x}{x^2 + 5}$$

$$\int_1^2 \frac{12x^2 + 20}{x(x^2 + 5)}\,dx = \int_1^2 \frac{4}{x} + \frac{8x}{x^2 + 5}\,dx$$

$$= \int_1^2 \frac{4}{x}\,dx + 4\int_1^2 \frac{2x}{x^2 + 5}\,dx$$

$$= \left[4\ln x + 4\ln(x^2 + 5)\right]_1^2 \qquad \textbf{1,1}$$

$$= 4\left[\ln x(x^2 + 5)\right]_1^2 = 4\left[\ln 18 - \ln 6\right] \qquad \textbf{1}$$

$$= 4\ln 3 (\approx 4{\cdot}39)$$

5.

$$xy^2 + 3x^2y = 4$$

$$y^2 + 2xy\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} = 0 \qquad \textbf{1M,1}$$

$$(2xy + 3x^2)\frac{dy}{dx} = -y^2 - 6xy$$

$$\frac{dy}{dx} = \frac{-y^2 - 6xy}{2xy + 3x^2} \qquad \textbf{1}$$

When $x = 1$,

$$y^2 + 3y = 4 \Rightarrow y^2 + 3y - 4 = 0 \Rightarrow (y + 4)(y - 1) = 0$$

$$\Rightarrow y = 1 \text{ since } y > 0 \qquad \textbf{1}$$

Hence at $(1,\ 1)$

$$\frac{dy}{dx} = \frac{-7}{5} \qquad \textbf{1}$$

Tangent is

$$(y - 1) = -\tfrac{7}{5}(x - 1) \qquad \textbf{1}$$

$$5y + 7x = 12.$$

Alternative for the first 3 marks.

$$xy^2 + 3x^2y = 4$$

$$y^2 + 3xy = \frac{4}{x}$$

$$2y\frac{dy}{dx} + 3y + 3x\frac{dy}{dy} = -\frac{4}{x^2} \qquad \textbf{1}$$

$$(2y + 3x)\frac{dy}{dx} = -\frac{4}{x^2} - 3y \qquad \textbf{1}$$

$$\frac{dy}{dx} = \frac{-\frac{4}{x^2} - 3y}{2y + 3x} \qquad \textbf{1}$$

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6. (a)

$$\det\begin{pmatrix}1 & x\\ x & 4\end{pmatrix} = 4 - x^2 \qquad \textbf{1}$$

A matrix is singular when its determinant is 0, hence $x = \pm 2$. **1**

(b) When $x = 2$, $A = \begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$

$$A^2 = \begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix} = \begin{pmatrix}5 & 10\\ 10 & 20\end{pmatrix} = 5A. \qquad \textbf{1}$$

$$A^4 = (A^2)^2 = (5A)^2 = 25A^2 = 125A. \qquad \textbf{2E1}$$

Evaluating $A^4 = \begin{pmatrix}125 & 250\\ 250 & 500\end{pmatrix} = 125A$ was accepted.

7.

$$\int 8x^2 \sin 4x\,dx = 8x^2 \int \sin 4x\,dx - \int 16x\left(\int \sin 4x\,dx\right)dx \qquad \textbf{1M,1}$$

$$= 8x^2\left(\frac{-1}{4}\cos 4x\right) - \int 16x \times \frac{-1}{4}\cos 4x\,dx \qquad \textbf{1}$$

$$= -2x^2 \cos 4x + 4\left[x \int \cos 4x\,dx - \int \frac{1}{4}\sin 4x\,dx\right] \qquad \textbf{1}$$

$$= -2x^2 \cos 4x + x \sin 4x + \tfrac{1}{4}\cos 4x + c \qquad \textbf{1}$$

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8. The rth term is

$$\binom{10}{r}(x^2)^{10-r}\left(\frac{1}{x}\right)^r$$ **1 for form**

1 for powers

$$= \binom{10}{r}x^{20-3r}$$ **1 for simplifying**

$$20 - 3r = 14 \Rightarrow r = 2 \qquad \textbf{1}$$

$$\text{term is } 45x^{14} \qquad \textbf{1}$$

or

$$\binom{10}{r}(x^2)^r\left(\frac{1}{x}\right)^{10-r} = \binom{10}{r}x^{3r-10} \qquad \textbf{1,1,1}$$

$$3r - 10 = 14 \Rightarrow r = 8 \qquad \textbf{1}$$

$$\text{term is } 45x^{14} \qquad \textbf{1}$$

9.

$$\frac{d}{dx}(\tan x) = \sec^2 x. \qquad \textbf{1}$$

$$1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x. \qquad \textbf{1}$$

$$\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx \qquad \textbf{1}$$

$$= \tan x - x + c \qquad \textbf{1}$$

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10. (a)
$$x'(t) = t^3 - 12t^2 + 32t$$
$$x''(t) = 3t^2 - 24t + 32$$
which is 32 when $t = 0$. **1**

(b)
$$x'(t) = t^3 - 12t^2 + 32t$$
$$\Rightarrow x(t) = \tfrac{1}{4}t^4 - 4t^3 + 16t^2 + c \quad \mathbf{1}$$
Since $x(0) = 0$, $c = 0$, so at time t
$$x(t) = \tfrac{1}{4}t^4 - 4t^3 + 16t^2. \quad \mathbf{1}$$
At O, $x = 0$
$$\tfrac{1}{4}t^4 - 4t^3 + 16t^2 = 0$$
$$t^2\left(t^2 - 16t + 64\right) = 0$$
$$t^2(t - 8)^2 = 0 \quad \mathbf{1}$$
The body returns to O when $t = 8$. **1**

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11. (a) Counter example $m = 2$. **1,1**
So statement is false.

(b) Let the numbers be $2n + 1$ and $2m$ then **1M**
$$(2n + 1)^3 + (2m)^2 = 8n^3 + 12n^2 + 6n + 1 + 4m^2 \quad \mathbf{1}$$
$$= 2\left(4n^3 + 6n^2 + 3n + 2m^2\right) + 1 \quad \mathbf{1}$$
which is odd.

OR

Proving algebraically that either the cube of an odd number is odd or the square of an even number is even. **1**
Odd cubed is odd and even squared is even. **1**
So the sum of them is odd. **1**

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12. $f(x) = x\ln(2 + x)$ so
$$f'(x) = \frac{x}{2+x} + \ln(2+x), f''(x) = \frac{2}{(2+x)^2} + \frac{1}{2+x},$$
$$f'''(x) = -\frac{4}{(2+x)^3} - \frac{1}{(2+x)^2} \quad \mathbf{1}$$
and so $f'(0) = \ln 2$, $f''(0) = 1$, $f'''(0) = -\frac{3}{4}$. **1**
$$\text{Thus } f(x) = (\ln 2)x + \frac{x^2}{2} - \frac{x^3}{8} + \ldots \quad \mathbf{1}$$
$$x\ln(2 - x) = -f(-x) \quad \mathbf{1}$$
$$= (\ln 2)x - \frac{x^2}{2} - \frac{x^3}{8} + \ldots \quad \mathbf{1}$$
$$x\ln\left(4 - x^2\right) = x\ln(2 + x) + x\ln(2 - x) \quad \mathbf{1}$$
$$= (2\ln 2)x - \frac{x^3}{4} + \ldots \quad \mathbf{1}$$
Alternative for first three marks:
$$f(x) = x\ln(2 + x) = x\left(\ln 2 + \ln\left(1 + \tfrac{x}{2}\right)\right) \quad \mathbf{1}$$
$$= x\left(\ln 2 + \tfrac{x}{2} - \tfrac{x^2}{8} + \ldots\right) \quad \mathbf{1}$$
$$= x\ln 2 + \tfrac{x^2}{2} - \tfrac{x^3}{8} + \ldots \quad \mathbf{1}$$

13.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2$$

$$m^2 - 3m + 2 = 0 \qquad \mathbf{1}$$

$$(m - 1)(m - 2) = 0$$

$$m = 1 \text{ or } m = 2 \qquad \mathbf{1}$$

Complementary function: $y = Ae^x + Be^{2x}$ **1**

For particular integral try $y = ax^2 + bx + c$ **1M**

$$\Rightarrow \frac{dy}{dx} = 2ax + b;\ \frac{d^2y}{dx^2} = 2a$$

Hence require

$$2a - 3(2ax + b) + 2(ax^2 + bx + c) = 2x^2 \qquad \mathbf{1}$$

$$2ax^2 + (-6a + 2b)x + (2a - 3b + 2c) = 2x^2$$

$$\Rightarrow a = 1;\ b = 3;\ c = \frac{7}{2} \qquad \mathbf{1}$$

General solution is: $y = Ae^x + Be^{2x} + x^2 + 3x + \frac{7}{2}$ **1**

When $x = 0, y = \frac{1}{2}$ and $\frac{dy}{dx} = 1.$

$$\tfrac{1}{2} = A + B + \tfrac{7}{2} \Rightarrow A + B = -3 \qquad \mathbf{1}$$

$$\tfrac{dy}{dx} = Ae^x + 2Be^{2x} + 2x + 3 \Rightarrow 1 = A + 2B + 3 \Rightarrow A + 2B = -2 \qquad \mathbf{1}$$

$$B = 1 \qquad A = -4 \qquad \mathbf{1}$$

Particular solution is

$$y = -4e^x + e^{2x} + x^2 + 3x + \tfrac{7}{2}.$$

14. (a)

$$\overrightarrow{AB} = \mathbf{i} - 2\mathbf{j} \qquad \overrightarrow{AC} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \qquad \mathbf{1}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ -1 & 2 & 2 \end{vmatrix} = (-4 - 0)\,\mathbf{i} - (2 - 0)\,\mathbf{j} + (2 - 2)\,\mathbf{k} \qquad \mathbf{1}$$

$$= -4\mathbf{i} - 2\mathbf{j}$$

Equation is

$$-4x - 2y = k$$

$$= -4(1) - 2(1) = -6 \qquad \mathbf{1}$$

$$\text{i.e. } -2x - y = -3$$

$$2x + y = 3$$

(b) In π_1: $2 \times 0 + a = 3 \Rightarrow a = 3.$ **1**

In π_2: $0 + 3a - b = 2 \Rightarrow b = 3a - 2 = 7.$ **1**

Hence the point of intersection is $(0,\ 3,\ 7)$.

Line of intersection: direction from

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -2 & 0 \\ 1 & 3 & -1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} - 10\mathbf{k} \qquad \mathbf{1}$$

$$x = 0 + 2t;\ y = 3 - 4t;\ z = 7 - 10t \qquad \mathbf{1}$$

There are many valid variations on this (including symmetric form) and these were marked on their merits.

(c) Let the angle be θ, then

$$\cos\theta = \left|\frac{(-4\mathbf{i} - 2\mathbf{j})\,.\,(\mathbf{i} + 3\mathbf{j} - \mathbf{k})}{\sqrt{4^2 + 2^2}\sqrt{1^2 + 3^2 + 1^2}}\right| = \left|\frac{-4 - 6}{\sqrt{20 \times 11}}\right| = \frac{5}{\sqrt{55}} \qquad \mathbf{1M, 1}$$

or

$$\sin\theta = \left|\frac{(-4\mathbf{i} - 2\mathbf{j}) \times (\mathbf{i} + 3\mathbf{j} - \mathbf{k})}{\sqrt{4^2 + 2^2}\sqrt{1^2 + 3^2 + 1^2}}\right| \qquad \mathbf{1M}$$

$$= \frac{\sqrt{2^2 + 4^2 + 10^2}}{\sqrt{20}\sqrt{11}} = \sqrt{\frac{120}{20 \times 11}} = \sqrt{\frac{6}{11}} \qquad \mathbf{1}$$

Hence $\theta \approx 47.6°$. **1**

15. (a) $$\frac{d}{dx}\left(\frac{x}{\ln x}\right) = \frac{1 \times \ln x - x \times \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}.$$ **1M,1**

$$\frac{d^2}{dx^2}\left(\frac{x}{\ln x}\right) = \frac{\frac{1}{x} \times (\ln x)^2 - (\ln x - 1) \times \frac{2 \ln x}{x}}{(\ln x)^4}$$ **1**

$$= \frac{\ln x - 2 \ln x + 2}{x(\ln x)^3} = \frac{2 - \ln x}{x(\ln x)^3}$$ **1**

(b) Stationary points when $\ln x = 1$, giving $x = e$ and $y = e$. **1**

At (e, e), the second derivative is

$$\frac{2 - 1}{e \times 1^3} > 0$$ **1**

so (e, e) is a minimum. **1**

(c) When $\frac{d^2y}{dx^2} = 0$, $\ln x = 2 \Rightarrow x = e^2$. **1**

$x = e^2 \Rightarrow y = \frac{1}{2}e^2$. **1**

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16. $z^k = \cos k\theta + i \sin k\theta$, **1**

so $$\frac{1}{z^k} = \frac{1}{\cos k\theta + i \sin k\theta} = \frac{\cos k\theta - i \sin k\theta}{\cos^2 k\theta + \sin^2 k\theta} = \cos k\theta - i \sin k\theta.$$ **2E1**

Adding the expressions for z^k and $\frac{1}{z^k}$ gives $z^k + \frac{1}{z^k} = 2 \cos k\theta$ so

$\cos k\theta = \frac{1}{2}\left(z^k + z^{-k}\right)$. **1**

Subtracting the expressions for z^k and $\frac{1}{z^k}$ gives $z^k - \frac{1}{z^k} = 2i \sin k\theta$ so

$\sin k\theta = \frac{1}{2i}\left(z^k - z^{-k}\right)$. **1**

For $k = 1$

$$\cos^2\theta \sin^2\theta = (\cos\theta \sin\theta)^2$$ **1**

$$= \left(\frac{\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right)}{4i}\right)^2$$ **2E1**

$$= -\frac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2.$$

$$\left(z^2 - \frac{1}{z^2}\right)^2 = z^4 + \frac{1}{z^4} - 2 = 2 \cos 4\theta - 2$$ **1**

$$\Rightarrow \cos^2\theta \sin^2\theta = \frac{1}{8} - \frac{1}{8}\cos 4\theta,$$ **1**

i.e. $a = \frac{1}{8}$ and $b = \frac{1}{8}$.

OR

A correct trigonometric proof that $\cos^2\theta \sin^2\theta = \frac{1}{8} - \frac{1}{8}\cos 4\theta$. **1**

[END OF MARKING INSTRUCTIONS]