# 2014 Mathematics

# Advanced Higher

# Finalised Marking Instructions

**Part One: General Marking Principles for Mathematics Advanced Higher**

This information is provided to help you understand the general principles you must apply when marking candidate responses to questions in this Paper. These principles must be read in conjunction with the specific Marking Instructions for each question.

**(a)** Marks for each candidate response must <u>always</u> be assigned in line with these general marking principles and the specific Marking Instructions for the relevant question.

**(b)** Marking should always be positive ie, marks should be awarded for what is correct and not deducted for errors or omissions.

**GENERAL MARKING ADVICE: Mathematics Advanced Higher**

The marking schemes are written to assist in determining the "minimal acceptable answer" rather than listing every possible correct and incorrect answer. The following notes are offered to support Markers in making judgements on candidates' evidence and apply to marking both end of unit assessments and course assessments.

**General Marking Principles**

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

**1** The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

**2** The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

**3** The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values/algebraic expressions.

**4** Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

**5** Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

**6** Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. When marking, no comments at all should be made on the script. The total mark for each question should appear in one of the right-hand margins. The following codes should be used where applicable:

✓ – correct; X – wrong; working underlined or circled – wrong;

tickcross – mark(s) awarded for follow-through from previous answer;

^ ^ - mark(s) lost through omission of essential working or incomplete answer;

wavy or broken underline – bad form, but not penalised.

**Part Two: Marking Instructions for each Question**

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **1.** | **(a)** | | $f'(x)=\frac{(x^2+1).2x-(x^2-1).2x}{(x^2+1)^2}$<br><br>$=\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$<br><br>$\boldsymbol{=\frac{4x}{(x^2+1)^2}}$<br><br>**OR**<br>$f(x)=1-\frac{2}{x^2+1}$<br><br>$f'(x)=-1(-2)(x^2+1)^{-2}...$<br><br>$...\times 2x$<br>$\therefore f'(x)=4x(x^2+1)^{-2}$<br>$=\frac{4x}{(x^2+1)^2}$ | **3** | •¹ Knows to use quotient (or product) rule.[1,2]<br>•² Correct derivative, using either rule, unsimplified.<br><br>•³ Simplifies to answer.<br><br>•¹ By polynomial division (or inspection) correctly simplifies *f*(*x*).<br>•² Correctly completes first step in integration.<br><br>•³ Applies chain rule *and* simplifies to answer. |
| **1.** | **(b)** | | $=\frac{6x}{1+(3x^2)^2}$<br><br>$\boldsymbol{=\frac{6x}{1+9x^4}}$<br><br>**OR**<br>$\tan y=3x^2$<br><br>$\sec^2 y.\frac{dy}{dx}=6x$<br><br>$\frac{dy}{dx}=\frac{6x}{\sec^2 y}=\frac{6x}{\sec^2\left(\tan^{-1}(3x^2)\right)}$ | **3** | •¹ Correct form of denominator.<br>•² Multiplies by $\frac{d}{dx}(3x^2)$<br>•³ Processes to remove brackets correctly.<br><br>•¹ Correctly processes from $\tan^{-1}$ to tan.<br>•² Correctly differentiates implicitly on both sides<br>•³ Isolates $\frac{dy}{dx}$ on LHS and expresses in terms of *x* only. |

**Notes:**

1.1 Evidence of method: Statement of the rule and evidence of progress in applying it.
**OR** Application showing the ***difference*** of two terms, both involving differentiation and a denominator.

1.2 Accept use of product use with equivalent criteria for •¹.

<table>
<tr><th colspan="3">Question</th><th>Expected Answer/s</th><th>Max Mark</th><th>Additional Guidance</th></tr>
<tr><td>2.</td><td></td><td></td><td>

$$=\binom{10}{r}\left(\frac{2}{x}\right)^{r}\left(\frac{1}{4x^2}\right)^{10-r} \quad \textbf{OR} \quad \binom{10}{r}\left(\frac{2}{x}\right)^{10-r}\left(\frac{1}{4x^2}\right)^{r}$$

$$=\binom{10}{r}\frac{2^{r}}{x^{r}\left(4x^2\right)^{10-r}} \quad \textbf{OR} \quad \binom{10}{r}\frac{2^{10-r}}{x^{10-r}\left(4x^2\right)^{r}}$$

$$=\binom{10}{r}\frac{2^{r}}{x^{r}4^{10-r}\left(x^2\right)^{10-r}} \quad \textbf{OR} \quad \binom{10}{r}\frac{2^{10-r}}{x^{10-r}4^{r}\left(x^2\right)^{r}}$$

$$=\binom{10}{r}\frac{2^{r}}{x^{r}2^{20-2r}x^{20-2r}} \quad \textbf{OR} \quad \binom{10}{r}\frac{2^{10-r}}{x^{10-r}2^{2r}x^{2r}}$$

$$=\binom{10}{r}\frac{2^{3r-20}}{x^{20-r}} \quad \textbf{OR} \quad \binom{10}{r}\frac{2^{10-3r}}{x^{10+r}}$$

$$=\binom{10}{r}2^{3r-20}x^{r-20} \quad \textbf{OR} \quad \binom{10}{r}2^{10-3r}x^{-r-10}$$

For term in $x^{-13}$: $r = 7$ **OR** $r = 3$

$$\text{ie } =\binom{10}{7}2^{3\times7-20}x^{7-20} \quad \textbf{OR} \quad \binom{10}{3}2^{10-3\times3}x^{-3-10}$$

$$\mathbf{= 240x^{-13}} \quad \textbf{OR} \quad \mathbf{\frac{240}{x^{13}}}$$

</td><td>5</td><td>

•¹ Unsimplified form of general term, correct (either form).[1]

•² Correct simplification of coefficients.[2,3]

•³ Correct simplification of indices of $x$.[2,3]

•⁴ Obtains appropriate value for $r$ from simplified expression.[4,5]

•⁵ Correct evaluation of above expression.[5]

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**Notes:**

2.1 No simplification required, but must be stated explicitly, as required in the question.

2.2 Negative indices may be written in denominator with positive indices.

2·3 Coefficients must be collected to a single expression. eg separate powers of both 4 and 2 or multiple powers of 2 are not permitted.

2·4 Where an incorrect, simplified expression leads to a non-integer value for $r$, •⁴ is not available.

2·5 Final answer obtained from expansion, with no general term, only •⁴ and •⁵ available [max 2 out of 5].

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| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **3.** | | | $\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 4 & 3 & -\lambda & 4 \\ 5 & 6 & 8 & 11 \end{array}\right)$ $\begin{array}{l} R_1 \\ 4R_1 - 1R_2 \\ R_3 - 5R_1 \end{array}$<br><br>$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 4+\lambda & 4 \\ 0 & 1 & 3 & 1 \end{array}\right)$ $\begin{array}{l} R_1 \\ R_2 \\ R_2 - R_3 \end{array}$<br><br>$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 4+\lambda & 4 \\ 0 & 0 & 1+\lambda & 3 \end{array}\right)$<br><br>$(1+\lambda)z = 3$<br><br>$z = \mathbf{\dfrac{3}{1+\lambda}}$<br><br>$\boldsymbol{\lambda \neq -1}$<br><br>$\mathbf{z = 1, y = -2, x = 3}$. | **6** | •[1] Sets up augmented matrix.[1]<br><br>•[2] Correctly obtains zeroes in first elements of second and third rows.[1,2]<br><br>•[3] Completes elimination to upper triangular form.[3]<br><br>•[4] Obtains simplified expression for *z*.[4,5]<br><br>•[5] Correct statement based on expression at •[4,6]<br>•[6] Correct solution based on matrix at •[3]. |

**Notes:**

3.1 Row operations commentary not required for full credit. Methods not using augmented matrix may be acceptable.

3.2 Not necessary to have unitary values for second elements for •[2].

3.3 Accept lower triangular form.

3.4 If lower triangular form used, will need to have simplified expression for *z*.

3.5 Accept $z = \dfrac{-3}{-1-\lambda}$.

3.6 Also accept: When $\lambda = -1$there are no solutions; $\lambda < -1$ and $\lambda > -1$; $\lambda < -1$ or $\lambda > -1$.

3.7 Do NOT accept: $-1 < \lambda > -1$.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| 4. | | | $\dfrac{dx}{dt}=\dfrac{2t}{1+t^2}$ | 3 | •[1] Correct differentiation of either $y$ or $x$ **OR** evidence of knowing to differentiate both equations w.r.t. $t$.[1] |
| | | | $\dfrac{dy}{dt}=\dfrac{4t}{1+2t^2}$ | | •[2] Correct completion of both differentiations. |
| | | | $\dfrac{dy}{dx}=\dfrac{4t\left(1+t^2\right)}{2t\left(1+2t^2\right)}$ | | |
| | | | $\dfrac{dy}{dx}=\dfrac{\mathbf{2\left(1+t^2\right)}}{\mathbf{\left(1+2t^2\right)}}$ **OR** $\dfrac{dy}{dx}=\dfrac{\mathbf{2+2t^2}}{\mathbf{1+2t^2}}$ | | •[3] Processes to answer.[2,3] |

**Notes:**

4.1 For example $\dfrac{dx}{dt}=\dfrac{1}{1+t^2}$ *and* $\dfrac{dy}{dt}=\dfrac{1}{1+2t^2}$.

4.2 Although $t \neq 0$ applies, do not penalise omission.

4.3 Expressed as a single fraction.

4.4 Failure to employ chain rule renders •[1] unavailable, but 2 out of 3 is possible for $\dfrac{\left(1+t^2\right)}{\left(1+2t^2\right)}$.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| 5. | | | $\begin{vmatrix} i & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$ | 3 | •1 Setting up cross product correctly. [1,4] |
| | | | $= -13\boldsymbol{i} + 5\boldsymbol{j} + 7\boldsymbol{k}$ | | •2 Correctly evaluates cross product. |
| | | | $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w}) = \begin{pmatrix} 5 \\ 13 \\ 0 \end{pmatrix}.\begin{pmatrix} -13 \\ 5 \\ 7 \end{pmatrix} = 0$ | | •3 Correctly evaluates dot product with ***u*** and vector from answer •2. |
| | | | ***u*** lies in the same plane as the one containing both ***v*** and ***w***.<br>**OR** ***u*** is parallel to the plane containing ***v*** and ***w***.<br>**OR** ***u*** is perpendicular to the normal of ***v*** and ***w***.<br>**OR** All **4** points lie in the same plane.<br>**OR** ***u*** is perpendicular to ***v*** × ***w***.<br>**OR** Volume of parallelepiped is zero.<br>**OR** ***u***, ***v*** and ***w*** are coplanar/linearly dependent. | 1 | •4 Any one correct statement. [2,3,6] |
| | | | **OR** $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} 5 & 13 & 0 \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$ | | •1 Setting up combined product correctly. [4] |
| | | | $= 5\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} - 13\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$ | | •2 Correctly processes determinant.[5] |
| | | | $= 0$ | | •3 Correctly evaluates determinant to reach 0. |

**Notes:**

5.1 Alternative layouts and methods possible for full credit.

5.2 Do NOT accept: Vectors are perpendicular [must specify which vectors].

5.3 •4 only available where statement is consistent with •3.

5.4 Rows 2 and 3 are not interchangeable.

5.5 This line of this version may be omitted with •2 and •3 awarded for final answer of 0.

5.6 Where incorrect answer at •3 is some $k \neq 0$, accept: Volume of parallelepiped = $k$ or negation of any one of the other statements.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| 6. | | | $y = 1\text{n}(x^3\cos^2 x)$<br>$y = 1\text{n}(x^3) + 1\text{n}(\cos^2 x)$<br>$\frac{dy}{dx} = \frac{3}{x} - \frac{2\sin x}{\cos x}$<br>$\frac{dy}{dx} = \frac{3}{x} - 2\tan x$<br>**a = 3, b = −2.**<br>– – – – – – – –<br>**OR**<br>$y = 1\text{n}(x^3\cos^2 x)$<br>$\frac{dy}{dx} = \frac{3x^2\cos^2 x - 2x^3 \sin x \cos x}{x^3\cos^2 x}$<br>$\frac{dy}{dx} = \frac{3}{x} - 2\tan x$<br>**a = 3, b = −2.**<br>– – – – – – – –<br>**OR**<br>$e^y \frac{dy}{dx} = 3x^2\cos^2 x - 2x^3 \sin x \cos x$<br>$\frac{dy}{dx} = \frac{3x^2\cos^2 x - 2x^3 \sin x \cos x}{x^3\cos^2 x}$<br>$\frac{dy}{dx} = \frac{3}{x} - 2\tan x$<br>**a = 3, b = −2.** | **3** | •[1] Correctly takes logs *and* correctly separates RHS terms.<br>•[2] LHS *and* either term of derivative on RHS correct.<br>•[3] Rest of derivative correct *and* values of *a* and *b*.<br>– – – – – – – –<br>•[1] Correctly takes logs *and* evidence of: $\frac{d}{dx}\left(1\text{n}\left[f(x)\right]\right) = \frac{f'(x)}{f(x)}$.<br>•[2] Completes differentiation correctly (unsimplified).<br>•[3] Simplifies to obtain correct form *and* values of *a* and *b*.<br>– – – – – – – –<br>•[1] Evidence of implicit differentiation.<br>•[2] Completes differentiation correctly.<br>•[3] Divides by $e^y$ correctly *and* simplifies to obtain correct form *and* values of *a* and *b*. |

**Notes:**

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **7.** | | | For $n = 1$ RHS $= \begin{pmatrix} 2^1 & a\left(2^1 - 1\right) \\ 0 & 1 \end{pmatrix}$ <br> $= \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ <br> $= A$ <br><br> LHS $= A^1 = A =$ RHS. <br> Assume true for $n = k$, <br> $A^k = \begin{pmatrix} 2^k & a\left(2^k - 1\right) \\ 0 & 1 \end{pmatrix}$ <br> Consider $n = k + 1$, <br> $A^{k+1} = A^k A^1$ [**OR** $A^1 A^k$] <br><br> $= \begin{pmatrix} 2^k & a\left(2^k - 1\right) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ <br> $= \begin{pmatrix} 2^k.2 & 2^k.a + a\left(2^k - 1\right) \\ 0 & 1 \end{pmatrix}$ <br> $= \begin{pmatrix} 2^{k+1} & 2^k.a + 2^k.a - a \\ 0 & 1 \end{pmatrix}$ <br> $= \begin{pmatrix} 2^{k+1} & a\left(2^k + 2^k - 1\right) \\ 0 & 1 \end{pmatrix}$ <br> $= \begin{pmatrix} 2^{k+1} & a\left(2^{k+1} - 1\right) \\ 0 & 1 \end{pmatrix}$ <br><br> Hence, if true for $n = k$, then true for $n = k + 1$, but since true for $n = 1$, then by induction true for all positive integers $n$. | **4** | •[1] Substituting $n = 1$.[1] <br><br> See note 5. <br><br> •[2] Inductive hypothesis (must include "Assume true for $n = k$…" or equivalent phrase) *and* expansion of. $A^{k+1}$.[2,5] <br><br> •[3] Correct multiplication of two matrices *and* accurate manipulation of indices and brackets.[3] <br><br> * <br><br> •[4] Line * *and* statement of result in terms of $(k + 1)$ *and* valid statement of conclusion.[4,6] |

**Question 7 Notes:**

7.1 Correct substitution necessary for $\bullet^1$. Accept starting with $A^1$ and proceeding via $\begin{pmatrix} 2^1 & a(2^1-1) \\ 0 & 1 \end{pmatrix}$ for $\bullet^1$.

7.2 Acceptable phrases include: "If true for..."; "Suppose true for…"; "Assume true for…". However, ***not*** acceptable: "Consider $n=k$" and "True for $n=k$".

7.3 No access to $\bullet^3$ or $\bullet^4$ without correct matrix multiplication. This includes any evidence that $2^k.2 = 4^k$ whether subsequently "corrected" or not, loses both $\bullet^3$ (if occurring before line $*$) and $\bullet^4$.

7.4 Minimum acceptable form for $\bullet^4$: "Then true for $n = k+1$, but since true for $n = 1$, then true for all $n$" or equivalent.

7.5 This expression (or equivalent) must appear somewhere for the award of $\bullet^2$. However, it may appear in later working.

7.6 Need meaningful prior working for award of $\bullet^4$.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **8.** | | | $4m^2 - 4m + 1 = 0$<br>$(2m - 1)^2 = 0$<br>$m = \frac{1}{2}$<br>C.F./G.S. $y = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x}$<br>$y = 4$ when $x = 0$ gives $4 = A.1+0$, so $A = 4$<br>$\frac{dy}{dx} = \frac{1}{2}Ae^{\frac{1}{2}x} + Be^{\frac{1}{2}x} + \frac{1}{2}Bxe^{\frac{1}{2}x}$<br>$\frac{dy}{dx} = 3$ when $x = 0$ gives<br>$3 = \frac{1}{2}Ae^0 + Be^0 + \frac{1}{2}B.0.e^0$<br>$3 = \frac{1}{2}.4 + B,$ so $B = 1$<br>So P.S. is: $\boldsymbol{y = 4e^{\frac{1}{2}x} + xe^{\frac{1}{2}x}}$ | **6** | •¹ Correct auxiliary equation.[1]<br>•² Correct solution of auxiliary equation.[4]<br>•³ Statement of general solution/complementary function.[3,4,5,6]<br>•⁴ Correct evaluation of *A*.[2,4]<br>•⁵ Correct differentiation of G.S.[4]<br>•⁶ Substitution to obtain B *and* particular solution.[4] |

**Notes:**

8.1 Or equivalent.

8.2 Accept calculation of *A* after differentiation.

8.3 Incorrectly using $y = Ae^{\frac{1}{2}x} + Be^{\frac{1}{2}x}$, but correctly carrying out (simplified) differentiation, leading to inconsistent equations: $A + B = 4$ and $A + B = 6$, gains •⁴ (for two equations). ie max 3 (out of 6).

8.4 Incorrect factorisation of auxiliary equation with real, distinct roots •² not awarded. Follow through marks available for •³, •⁴ and •⁶. Since working is eased, •⁵ not available. ie max 4 out of 6. When complex roots result, •⁵ IS available since working is certainly not eased. ie max 5 out of 6. Incorrect, equal roots max 5 out of 6 as working eased, but not significantly.

8.5 Starting at $y = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x}$ with no prior working loses •¹ and •².

8.6 However, if A.E. appears (i.e. $4m^2 + 4m + 1 = 0$) and jump straight to $y = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x}$ then award $\left[\frac{3}{3}\right]$.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| 9. | | | $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$<br><br>$\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} \ldots$<br>$= 1 - \frac{9x^2}{2} + \frac{81x^4}{24} \ldots$<br>$= 1 - \frac{9x^2}{2} + \frac{27x^4}{8} \ldots$ | 2 | •[1] Correct statement of series for cos $x$.[1,2]<br><br>•[2] Substitution and evaluation of coefficients.[3] |
| | | | **OR**<br>$f(x) = \cos 3x$ $\quad f(0) = 1$<br>$f'(x) = -3\sin 3x$ $\quad f'(0) = 0$<br>$f''(x) = -9\cos 3x$ $\quad f''(0) = -9$<br>$f'''(x) = 27\sin 3x$ $\quad f'''(0) = 0$<br>$f''''(x) = 81\cos 3x$ $\quad f''''(0) = 81$ | | •[1] Correct differentiation and evaluation if doing from first principles. |
| | | | $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots$<br>$= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \ldots$<br><br>$e^{2x}\cos 3x = \left(1 - \frac{9x^2}{2} + \frac{27x^4}{8} \ldots\right)\left(1 + 2x + 2x^2 + \frac{4x^3}{3} \ldots\right)$<br>$= 1 + 2x + 2x^2 + \frac{4x^3}{3} - \frac{9x^2}{2} - \frac{18x^3}{2} \ldots$<br><br>$= \mathbf{1 + 2x - \frac{5x^2}{2} - \frac{23x^3}{3}}$ | 1 | •[3] Correctly stating series with correct substitution.[3]<br><br>•[4] Knows to multiply the two previously obtained series together.<br><br>•[5] Correctly multiplies out brackets.[4]<br><br>•[6] Simplifies to lowest terms.[3,4] |

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| 9. | | | **OR**<br><br>$f(x)=e^{2x}\cos 3x$ $\quad f(0)=1$<br>$f'(x)=2e^{2x}\cos 3x-3e^{2x}\sin 3x$ $\quad f'(0)=2$<br>-------------------------------------------------------------------<br>$f''(x)=e^{2x}(-5\cos 3x-12\sin 3x)$ $\quad f''(0)=-5$<br>$f'''(x)=e^{2x}(-46\cos 3x-9\sin 3x)$ $\quad f'''(0)=-46$<br><br>$e^{2x}\cos 3x=1+2x+\frac{(-5)x^2}{2!}+\frac{(-46)x^3}{3!}\ldots$<br><br>$e^{2x}\cos 3x=1+2x-\frac{5x^2}{2}-\frac{46x^3}{6}\ldots$<br><br>$\mathbf{=1+2x-\frac{5x^2}{2}-\frac{23x^3}{3}}$ | | •4 Either all three derivatives correct **OR** first derivative *and* first two evaluations (above ------) **OR** all evaluations [not eased if at least one each of $e^{2x}$ and sin/cos$3x$ ] **OR** last two derivatives *and* last two evaluations correct.<br>•5 Remainder correct.<br><br>•6 Correct substitution of coefficients obtained at •4 into formula and simplifies to lowest terms.[4] |

**Notes:**

9.1 Award •1 for substitution of $3x$ into series for cos$x$.

9.2 Must have at least 3 terms for •1 if no further working.

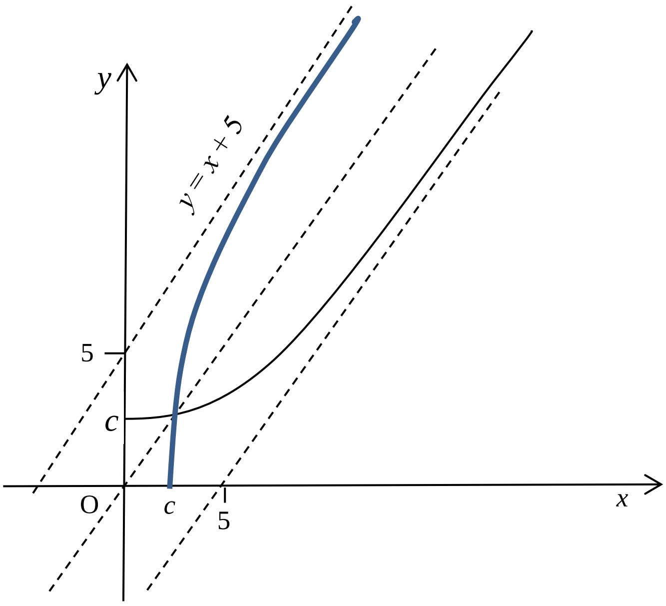
9.3 Candidates may differentiate from first principles for any or all of the three required series for full credit.

9.4 For •5 and •6 ignore additional terms in $x^4$ or higher.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **10.** | | | $(x-1)^2+y^2=4$<br><br>$V=\pi\int_0^3 y^2dx$<br><br>$=\pi\int_0^3\left(4-(x-1)^2\right)\,dx$<br><br>$=\pi\left[4x-\frac{1}{3}(x-1)^3\right]_0^3$<br><br>$=\mathbf{9\pi}$ units$^3$ | **5** | •[1] Correctly identifies circle equation.[1,5,7,8,11]<br>•[2] Correct form of integral *and* applies correct limits.[4,11]<br>•[3] Substitutes correct expression for $y^2$.[1,8,11]<br>•[4] Integrates function correctly.[8, 10,11]<br>•[5] Correctly evaluates expression.[2,6,8,9,11] |

**Notes:**

10.1 •[1] awarded for correct circle equation and if incorrectly manipulated thereafter, •[3] not awarded.
10.2 If •[4] awarded, may award •[5] for an approximate answer (28·3 or more accurate: 28·27433…).
10.3 Need to have a positive final value for volume to qualify for •[5].
10.4 *dx* essential.
10.5 May translate semi-circle one unit left without penalty, if done correctly.
10.6 Correct evaluation of any expression to a positive final answer earns •[5].
10.7 Accept any version of the equation of the circle.
10.8 **N.B.** several wrong methods still lead to 9π. Take care to ensure that the method used is valid.
10.9 Evaluations of expressions involving logs are most likely to go wrong (especially when missing absolute value signs) at some point and will be penalised. eg log −2 = log 2 loses •[5].
10.10 •[4] not available if working eased significantly, eg when integrating only a linear function to a quadratic function.
10.11 Halving value at any point or at end leading to $\frac{9}{2}\pi$ units$^3$ loses •[5].

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **11.** | **(a)** | |  | **4** | $\bullet^1$ Correct shape and behaviour approaching asymptote.<br><br>$\bullet^2$ Asymptote parallel to original.[2,3]<br><br>$\bullet^3$ 5 marked on $y$-axis at asymptote and $c$ on $x$-axis.[4]<br><br>$\bullet^4$ Symmetry.[1] |
| **11.** | **(b)** | | $y = x - 3$ | **1** | $\bullet^5$ Correct statement of equation of asymptote.[5] |
| **11.** | **(c)** | | From the diagram, the two **curves/graphs** intersect<br>**OR**<br>$y = f(x)$ intersects $y = x$<br>**OR**<br>$y = f^{-1}(x)$ intersects $y = x$<br>**OR**<br>$f^{-1}(x) = f(x)$<br>**So $x = f(f(x))$** | **1** | $\bullet^6$ Valid reason from observation of graph or algebraic.[6,7] |

**Note:**

11a.1 Possible to award $\bullet^4$ without $y = x$ on diagram.
11a.2 Where asymptotes meet or are clearly not parallel, do not award $\bullet^2$.
11a.3 Statement of equation of asymptote not necessary for $\bullet^2$ .
11a.4 Accept asymptote of inverse passing through $(-5,0)$ for $\bullet^3$ only if labelled as $y = x + 5$.
11b.5 Award mark for any form of straight line equation.
11c.6 "Two lines intersect" not sufficient for $\bullet^6$ unless lines referred to are specified.
11c.7 Where sketch indicates neither curve crossing $y = x$ or not crossing each other, do not award $\bullet^6$, even with an assertion of the form "$x = f(f(x))$ has no solutions" as this contradicts the statement in the question.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **12.** | | | $x = \tan\theta$<br>$\dfrac{dx}{d\theta} = \sec^2\theta$<br>$dx = \sec^2\theta d\theta$<br>$x = 1$ and $x = 0$ become $\theta = \dfrac{\pi}{4}$ and $\theta = 0$.<br>$\int_0^{\frac{\pi}{4}} \dfrac{\sec^2\theta d\theta}{\left(1+\tan^2\theta\right)^{\frac{3}{2}}}$<br>$\int_0^{\frac{\pi}{4}} \dfrac{\sec^2\theta d\theta}{\left(\sec^2\theta\right)^{\frac{3}{2}}}$<br>$\int_0^{\frac{\pi}{4}} \cos\theta d\theta$<br>$= \left[\sin\theta\right]_0^{\frac{\pi}{4}} = \dfrac{1}{\sqrt{2}}$<br>**OR** $x = \tan\theta$ so $\theta = \tan^{-1} x$<br>$d\theta = \dfrac{1}{1+x^2} dx$ so $dx = d\theta(1+x^2)$<br>$x = 1$ and $x = 0$ become $\theta = \dfrac{\pi}{4}$ and $\theta = 0$.<br>$\int_0^{\frac{\pi}{4}} \dfrac{(1+x^2)d\theta}{\left(1+x^2\right)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{4}} \dfrac{d\theta}{\left(1+x^2\right)^{\frac{1}{2}}} = \int_0^{\frac{\pi}{4}} \dfrac{d\theta}{\left(1+\tan^2\theta\right)^{\frac{1}{2}}}$ | **6** | •[1] Correctly differentiating substitution expression.<br>•[2] Processing substitution to obtain *both* limits for $\theta$.[1,4]<br>•[3] Correctly replacing all terms.<br>•[4] Replaces $1 + \tan^2\theta$ with $\sec^2\theta$.<br>•[5] Simplifies to integrable form.[4]<br>•[6] Integrates *and* evaluates correctly.[2,3,4]<br><br>•[1] Correctly differentiating rearranged expression.<br>•[2] Processing substitution to obtain *both* limits for $\theta$.[1,4]<br>•[3] Correctly replacing all terms and expressing in terms of $\theta$ only. |

**Notes:**

12.1 Full credit available to candidates progressing clearly with *x* limits and substituting to arrive at function of *x* and using *x* limits then.

12.2 Approximations are not acceptable since the exact value was specified.

12.3 Alternative exact values such as $\dfrac{\sqrt{2}}{2}$ should be awarded •[6].

12.4 Where candidate keeps limits as $x = 0$ & $x = 1$, may earn •[2] later by replacing $\sin\theta$ with $\dfrac{x}{\sqrt{1+x^2}}$, with or without right-angled triangle justification, without penalty.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **13.** | | | $\frac{dF}{dx} = 0 + e^x(\cos x + \sin x) + e^x(\sin x - \cos x - \sqrt{2})$ <br> $= e^x(2\sin x - \sqrt{2})$ <br> For S.P.s, $\frac{dF}{dx} = 0$, so $e^x(2\sin x - \sqrt{2}) = 0$ <br> Then $2\sin x = \sqrt{2}$ and so $\sin x = \frac{1}{\sqrt{2}}$ <br> Hence $x = \frac{\pi}{4}, \frac{3\pi}{4}$ <br> Leading to values of $F$ = 11·9, 15 <br> $40 \le s \le 120$ so $0 \le x \le \pi$ <br> $x = 0, F \simeq 12\cdot6;\ x = \pi, F \simeq 5\cdot4$ <br> **Greatest efficiency 15 km/litre at 100 km/h**. <br> **Least efficiency 5·4 km/litre at 120 km/h.** | **10** | •[1] Evidence of differentiation *and* one term correct. <br> •[2] A second correct term. <br> •[3] Third term correct. <br> •[4] Sets to 0 *and* solves to obtain value for sin*x*. <br> •[5] Correctly obtains two solutions for *x*. [2] <br> •[6] Then obtains the two related values for *F*. [2] <br> •[7] Identifies correct upper *and* lower limits for *x*. [1] <br> •[8] Correctly obtains values for *F* at *both* endpoints. [6] <br> •[9] Correctly worded statement for greatest or least efficiency, with units.[3,10] <br> •[10] Correctly worded statement for other extreme.[4,10] |

**Notes for question 13:**

13.1 Alternatively, award for evidence of inputting both 0 and $\pi$ into $F$ to establish endpoints.

13.2 For only one value for $x$ and the correctly obtained value for $F$, award $\bullet^6$ but not $\bullet^5$.

13.3 To qualify for $\bullet^9$, need to compare at least 3 **positive** values.

13.4 To qualify for $\bullet^{10}$, need to compare at least 4 **positive** values.

13.5 Award $\bullet^5$, where answers given in degrees.

13.6 Where candidate has used degrees, negative answers for $F$ are likely. In which case, no follow through mark available for $\bullet^8$.

13.7 Where a candidate has $\frac{\pi}{4}$ only solution for $x$, they will not be awarded $\bullet^5$ and $\bullet^{10}$ is not available. Max 8 out of 10. Where only solution is $x = \frac{3\pi}{4}$, loses $\bullet^5$ and $\bullet^9$ not available.

13.8 Where a candidate has two solutions for $x$, but in degrees, they will not be awarded $\bullet^6$. Also, $\bullet^9$ and $\bullet^{10}$ are not available. Maximum mark: 7 out of 10.

13.9 Where a candidate has only one solution for $x$, but in degrees, they will not be awarded $\bullet^5$ or $\bullet^6$. Also, $\bullet^9$ and $\bullet^{10}$ are not available. Maximum mark: 6 out of 10.

13.10 Appearance of units for both efficiency and speed in either (or both) of greatest and least statements necessary to achieve $\bullet^9$. Appropriate speed to be stated for award of $\bullet^9$ and $\bullet^{10}$.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **14.** | **(a)** | | $1+r+r^2+r^3+\ldots=\dfrac{1}{1-r}$ <br><br> $\dfrac{1}{2-3r}=\dfrac{1}{2\left(1-\dfrac{3r}{2}\right)}$ **OR** $\dfrac{1}{1-(3r-1)}$ **OR** $\dfrac{\frac{1}{2}}{1-\frac{3}{2}r}$ <br><br> $=\dfrac{1}{2}\left(\dfrac{1}{1-\dfrac{3r}{2}}\right)=\dfrac{1}{2}\left(1+\dfrac{3r}{2}+\left(\dfrac{3r}{2}\right)^2+\ldots\right)$ <br><br> $=\dfrac{1}{2}\left(1+\dfrac{3r}{2}+\dfrac{9r^2}{4}+..\right)$ <br><br> $\left\|\dfrac{3r}{2}\right\|<1, \quad \therefore \|r\|<\dfrac{2}{3}$ | **4** | •[1] Correct statement of sum. <br><br> •[2] Valid rearrangement of expression. [2, 5] <br><br> •[3] Makes correct substitution for $r$ in series at •[1]. [1,3,5] <br><br> •[4] Correct statement of range. [5] |

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **14.** | **(b)** | | $\frac{1}{3r^2-5r+2}=\frac{A}{(3r-2)}+\frac{B}{(r-1)}$<br><br>$\therefore A(r-1)+B(3r-2)\equiv 1; \quad B=1$<br>$A=-3$<br><br>$\frac{1}{3r^2-5r+2}=\frac{-3}{(3r-2)}+\frac{1}{(r-1)}$<br>$=\frac{3}{(2-3r)}-\frac{1}{(1-r)}$ | **6** | •5 Correct form of partial fractions.[7]<br>•6 Either coefficient correct.[7]<br>•7 Second coefficient correct.[7] |
| | | | $=3\left(\frac{1}{2}\left(1+\frac{3r}{2}+\frac{9r^2}{4}+...\right)\right)-\left(1+r+r^2+...\right)$<br>$=\frac{1}{2}+\frac{5r}{4}+\frac{19r^2}{8}...$ | | •8 Recognising form *and* manipulating correctly.<br>•9 Simplifying to obtain first three terms. |
| | | | $\left\lvert\frac{3r}{2}\right\rvert<1$ and $\lvert r\rvert<1$, so $\lvert r\rvert<\frac{2}{3}$ | | •10 Correct statement of range of convergence.[6] |
| | | | **OR**<br>$f(x)=(3r^2-5r+2)^{-1} \qquad f(0)=\frac{1}{2}$<br>$f'(x)=-(3r^2-5r+2)^{-2}(6r-5) \qquad f'(0)=\frac{5}{4}$<br>$f''(x)=-6(3r^2-5r+2)^{-2}+2(3r^2-5r+2)^{-3}(6r-5)^2$<br>$f''(0)=\frac{19}{4}$ | | •8 Either first two lines correct OR both derivatives correct OR evaluation of all three expressions.[4] |
| | | | $\therefore f(x)=\frac{1}{2}+\frac{5r}{4}+\frac{19r^2}{8}...$ | | •9 Completes to obtain series. |

**Note:**

14.1 First three terms required. Ignore any subsequent terms.

14.2 Other arrangements possible for full credit. However, some arrangements will have different ranges of convergence and/or need to exclude $r = 0$ to avoid division by 0.

14.3 Evaluation of coefficients must not be significantly eased, so requires evaluation of two terms in third line. •10 not available where Maclaurin used to obtain series in (b).

14.4 May award •1 when series obtained using Maclaurin's or binomial theorems, but not •2, •3 or •4.

14.5 Range must be consistent with fractions in •8. •10 not available if final range given is $\lvert r\rvert<1$.

14.6 Where denominator factorised incorrectly, follow-through marks for •6 and •7 still available.

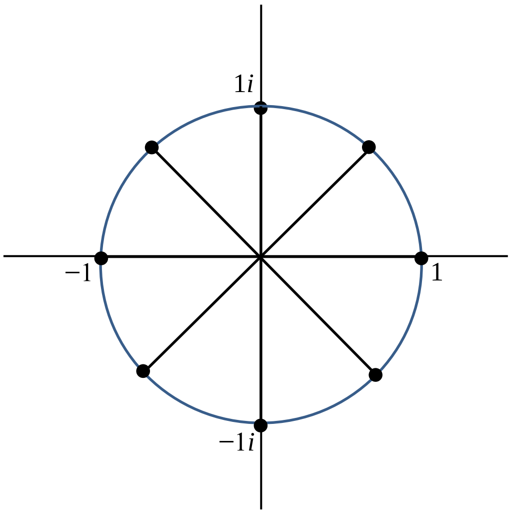
| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **15.** | **(a)** | | $\int e^x \cos x\, dx = e^x \cos x - \int e^x \left(\frac{d}{dx}(\cos x)\right) dx$<br>$= e^x \cos x + \int e^x \sin x\, dx$<br>$= e^x \cos x + e^x \sin x - \int e^x \cos x\, dx$<br><br>$\therefore 2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + c$<br>$\therefore \int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x) + c$ | **4** | •[1] Evidence of application of integration by parts. [4]<br>•[2] Completes first application.<br>•[3] Completes 2nd application.[3]<br><br>•[4] Recognises form of remaining integral and completes manipulation correctly. |
| | | | --------------------------------------------------------------------<br>**OR**<br>$\int e^x \cos x\, dx = e^x \sin x - \int e^x \left(\int \cos x\, dx\right) dx$<br>$= e^x \sin x - \int e^x \sin x\, dx$<br>$= e^x \sin x - (-e^x \cos x - \int -e^x \cos x\, dx)$<br><br>$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$<br>$\therefore 2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + c$<br>$\therefore \int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x) + c$ | | -----------------------------------<br>•[1] Evidence of application of integration by parts. [4]<br>•[2] Completes first application.<br>•[3] Completes 2nd application.[3]<br><br>•[4] Recognises form of remaining integral and completes manipulation correctly. |

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| 15. | (b) | | $I_n = e^x \cos nx - \int e^x(-n\sin nx)dx$<br>$= e^x \cos nx + \int e^x n\sin nx\,dx$<br>$= e^x \cos nx + e^x n\sin nx - \int n^2 e^x \cos nx\,dx$<br>$\left(1+n^2\right)I_n = ne^x \sin nx + e^x \cos nx \ +c$<br>$I_n = \left(\frac{e^x}{1+n^2}\right)\left(n\sin nx + \cos nx\right) \ +c$<br>-----------------------------------------------------------------<br>**OR**<br>$I_n = e^x \frac{1}{n}\sin nx - \int e^x \frac{1}{n}\sin nx\,dx$<br>$= \frac{1}{n}e^x \sin nx + \frac{1}{n^2}e^x \cos nx - \int \frac{1}{n^2}e^x \cos nx\,dx$<br>$\left(1+\frac{1}{n^2}\right)I_n = \frac{1}{n}e^x \sin nx + \frac{1}{n^2}e^x \cos nx \ +c$<br>$I_n = \left(\frac{1}{1+\frac{1}{n^2}}\right)\left(\frac{1}{n}e^x \sin nx + \frac{1}{n^2}e^x \cos nx\right) \ +c$<br>$I_n = \left(\frac{n^2}{n^2+1}\right)\left(\frac{1}{n}e^x \sin nx + \frac{1}{n^2}e^x \cos nx\right) +c$<br>$I_n = \left(\frac{e^x}{n^2+1}\right)\left(n\sin nx + \cos nx\right) +c$ | 4 | •[5] Completes first application on $I_n$.[4]<br>•[6] Completes second application on $I_n$.[3]<br>•[7] Correctly identifies and starts to manipulate $I_n$ terms.<br>•[8] Simplifies expression.<br>--------------------------------------------<br>•[5] Completes first application on $I_n$.[4]<br>•[6] Completes second application on $I_n$.[3]<br>•[7] Correctly identifies and starts to manipulate $I_n$ terms.<br>•[8] Simplifies expression. |

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| 15. | (c) | | $I_8 = \left[\left(\dfrac{e^x}{8^2+1}\right)(8\sin 8x + \cos 8x)\right]_0^{\frac{\pi}{2}}$ <br><br> $= \dfrac{1}{65}\left(e^{\frac{\pi}{2}} - 1\right)$ | 2 | •9 Correct substitution of value of $n$ into expression obtained in part (b) or equivalent expression.[5,6] <br><br> •10 Processes to statement of answer.[2,6] |

**Notes:**

15.1 Repeating errors such as $\int \sin x\, dx = \cos x$ should not be penalised twice. Award follow-through marks where appropriate.
15.2 Accept approximations to 3s.f. or better, ie 0·0586 (or better: 0·05862273…)
15.3 Not necessarily including simplification of + and – signs.
15.4 It may be that some candidates try both the given methods. In this case, mark positively, awarding marks in either portion, wherever the criteria for the marks are met.
15.5 Where expression from part (a) has been altered by inserting $n$ in one or more places, •9 is available for correct evaluation (subject to not being significantly eased – see note 15.6).
15.6 Where expression being evaluated is eased, then a correct evaluation will earn •10, but not •9. Consider anything containing all of: $e^x$, $\sin 8x$ *and* $\cos 8x$ as being of equivalent difficulty.

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **16.** | **(a)** | | $-1 = r(\cos\theta + i\sin\theta) = 1(\cos\pi + i\sin\pi)$<br>$= \cos\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)$<br>$\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$<br>$z = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right),$<br>$\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right), \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$<br>$z = \cos\left(\frac{\pi}{4}\right) \pm i\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) \pm i\sin\left(\frac{3\pi}{4}\right)$ | **3** | •[1] Polar form.[1]<br>•[2] Demonstrates understanding of method for 4th roots.<br>•[3] Obtains all four correct values.[2,3] |
| **16.** | **(b)** | | $z = \pm i,\ \pm 1$ | **2** | •[4] Any two solutions.[2]<br>•[5] Remaining two.[2] |
| **16.** | **(c)** | |  | **1** | •[6] Diagram showing all solutions to (a) and (b).[4] |

| Question | | | Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|---|---|---|
| **16.** | **(d)** | | $z^8 - 1 = (z^4 + 1)(z^4 - 1)$<br><br>Then the solutions to $z^4 + 1 = 0$ and $z^4 - 1 = 0$ are also the solutions to $z^8 - 1 = 0$. | **2** | •7 Factorises into two correct factors.<br><br>•8 Statement.5 |
| | **(e)** | | Observe that $z^6 + z^4 + z^2 + 1 = (z^2 + 1)(z^4 + 1)$<br>**OR**<br>$z^8 - 1 = (z^4 + 1)(z^2 + 1)(z^2 - 1)$<br><br>$\therefore$ Six solutions are those above except $z = \pm 1$ | **2** | •9 Factorises to obtain either form.<br><br>•10 Statement of solutions explicitly or as here. |

**Notes:**

16.1 Accept $\theta$ in degrees or radians. Polar form only.

16.2 Accept angles expressed in radians: $-\pi \le \theta \le \pi$ or $0 \le \theta \le 2\pi$ and degrees $-180° \le \theta \le 180°$ or $0° \le \theta \le 360°$.

16.3 Accept in Cartesian form: $\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$. Do not accept decimal approximations.

16.4 Argand diagram must include all candidate's solutions (at least 5 correct) as well as either one axis correctly scaled or all solutions labelled. Argand diagram need not include a circle or regular polygon.

16.5 Accept verification that all 8 answers from (a) and (b) index 8 = 1. For between four and seven solutions, all correctly verified: award •7, but not •8.

[END OF MARKING INSTRUCTIONS]