2002 Advanced Higher

1. Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x + y + 3z &= 2 \\ 2x + y + z &= 2 \\ 3x + 2y + 5z &= 5. \end{aligned}$$ 5

2. Verify that i is a solution of $z^4 + 4z^3 + 3z^2 + 4z + 2 = 0$.
Hence find all the solutions. 5

3. A curve is defined by the parametric equations

$$x = t^2 + t - 1, \quad y = 2t^2 - t + 2$$

for all t. Show that the point A $(-1, 5)$ lies on the curve and obtain an equation of the tangent to the curve at the point A. 6

4. (a) Given that $f(x) = \sqrt{x}\, e^{-x}$, $x \geq 0$, obtain and simplify $f'(x)$. 4

(b) Given $y = (x + 1)^2 (x + 2)^{-4}$ and $x > 0$, use logarithmic differentiation to show that $\frac{dy}{dx}$ can be expressed in the form $\left(\frac{a}{x+1} + \frac{b}{x+2}\right)y$, stating the values of the constants a and b. 3

5. Use integration by parts to evaluate $\int_0^1 \ln(1+x)\, dx$. 5

6. Use the substitution $x + 2 = 2\tan\theta$ to obtain $\int \frac{1}{x^2 + 4x + 8}\, dx$. 5

7. Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers n. 5

8. Express $\dfrac{x^2}{(x+1)^2}$ in the form $A + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$, $(x \neq -1)$, stating the values of the constants A, B and C. **3**

A curve is defined by $y = \dfrac{x^2}{(x+1)^2}$, $(x \neq -1)$.

(i) Write down equations for its asymptotes. **2**

(ii) Find the stationary point and justify its nature. **4**

(iii) Sketch the curve showing clearly the features found in (i) and (ii). **2**

9. Functions $x(t)$ and $y(t)$ satisfy

$$\frac{dx}{dt} = -x^2 y, \qquad \frac{dy}{dt} = -xy^2.$$

When $t = 0$, $x = 1$ and $y = 2$.

(*a*) Express $\dfrac{dy}{dx}$ in terms of x and y and hence obtain y as a function of x. **5**

(*b*) Deduce that $\dfrac{dx}{dt} = -2x^3$ and obtain x as a function of t for $t \geq 0$. **5**

10. Define $S_n(x)$ by

$$S_n(x) = 1 + 2x + 3x^2 + \ldots + nx^{n-1},$$

where n is a positive integer.

Express $S_n(1)$ in terms of n. **2**

By considering $(1 - x)\, S_n(x)$, show that

$$S_n(x) = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{(1-x)}, \quad x \neq 1.$$ **4**

Obtain the value of $\lim_{n\to\infty} \left\{ \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \ldots + \frac{n}{3^{n-1}} + \frac{3}{2}.\frac{n}{3^n} \right\}$. **3**

11. (*a*) Find an equation for the plane π_1 which contains the points $A(1, 1, 0)$, $B(3, 1, -1)$ and $C(2, 0, -3)$. **4**

(*b*) Given that π_2 is the plane whose equation is $x + 2y + z = 3$, calculate the size of the acute angle between the planes π_1 and π_2. **3**

12. A matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. Prove by induction that

$$A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix},$$

where n is any positive integer. **6**

13. Find the Maclaurin expansion of

$$f(x) = \ln(\cos x), \ \left(0 \le x < \tfrac{\pi}{2}\right),$$

as far as the term in x^4. **5**

14. Write down the 2×2 matrix A representing a reflection in the x-axis and the 2×2 matrix B representing an anti-clockwise rotation of $30°$ about the origin.

Hence show that the image of a point (x, y) under the transformation A followed by the transformation B is $\left(\frac{kx + y}{2}, \ \frac{x - ky}{2}\right)$, stating the value of k. **4**

15. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4\cos x.$$ **6**

Hence determine the solution which satisfies $y(0) = 0$ and $y'(0) = 1$. **4**

[END OF QUESTION PAPER]