X100/13/01

NATIONAL QUALIFICATIONS 2015

WEDNESDAY, 20 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1 Calculators may be used in this paper.

2 Candidates should answer **all** questions.

3 **Full credit will be given only where the solution contains appropriate working.**

PB

Marks

Answer all the questions

1. Use the binomial theorem to expand and simplify

$$\left(\frac{x^2}{3} - \frac{2}{x}\right)^5.$$

4

2. (*a*) For $y = \dfrac{5x+1}{x^2+2}$, find $\dfrac{dy}{dx}$. Express your answer as a single, simplified fraction. **3**

(*b*) Given $f(x) = e^{2x}\sin^2 3x$, obtain $f'(x)$. **3**

3. The sum of the first twenty terms of an arithmetic sequence is 320.

The twenty-first term is 37.

What is the sum of the first ten terms? **5**

4. The equation $x^4 + y^4 + 9x - 6y = 14$ defines a curve passing through the point A(1, 2).

Obtain the equation of the tangent to the curve at A. **4**

5. Obtain the value(s) of p for which the matrix $A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$ is singular. **4**

6. For $y = 3^{x^2}$, obtain $\dfrac{dy}{dx}$. **3**

7. Use the Euclidean algorithm to find integers p and q such that

$$3066p + 713q = 1.$$ **4**

Marks

8. Given $x = \sqrt{t+1}$ and $y = \cot t$, $0 < t < \pi$,

obtain $\frac{dy}{dx}$ in terms of t. **3**

9. Show that

$$\binom{n+2}{3} - \binom{n}{3} = n^2,$$

for all integers, n, where $n \geq 3$. **4**

10. Obtain the exact value of $\int_0^2 x^2 e^{4x} dx$. **5**

11. Write down the 2×2 matrix, M_1, associated with a reflection in the y-axis.

Write down a second 2×2 matrix, M_2, associated with an anticlockwise rotation through an angle of $\frac{\pi}{2}$ radians about the origin.

Find the 2×2 matrix, M_3, associated with an anticlockwise rotation through $\frac{\pi}{2}$ radians about the origin followed by a reflection in the y-axis.

What single transformation is associated with M_3? **4**

12. Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8. **3**

13. By writing z in the form $x + iy$:

(*a*) solve the equation $z^2 = |z|^2 - 4$; **3**

(*b*) find the solutions to the equation $z^2 = i(|z|^2 - 4)$. **4**

[Turn over

Marks

14. For some function, f, define

$$g(x) = f(x) + f(-x) \quad \text{and}$$
$$h(x) = f(x) - f(-x).$$

Show that $g(x)$ is an even function and that $h(x)$ is an odd function.

Hence show that $f(x)$ can be expressed as the sum of an even and an odd function. **4**

15. A line, L_1, passes through the point P(2, 4, 1) and is parallel to

$$\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and a second line, L_2, passes through Q(–5, 2, 5) and is parallel to

$$\mathbf{u}_2 = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

(*a*) Write down the vector equations for L_1 and L_2. **2**

(*b*) Show that the lines L_1 and L_2 intersect and find the point of intersection. **4**

(*c*) Determine the equation of the plane containing L_1 and L_2. **4**

16. Solve the second order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x}$$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 0$. **10**

17. Find $\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}\,dx$, $x > 3$. **9**

Marks

18. Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli's Law states that the rate of change of volume, V, of water in the tank is proportional to the square root of the height, h, of the water above the hole.

This is given by the differential equation:

$$\frac{dV}{dt} = -k\sqrt{h},\ k > 0.$$

(*a*) For a cylindrical tank with constant cross-sectional area, A, show that the rate of change of the height of the water in the tank is given by

$$\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}.$$ **2**

(*b*) Initially, when the height of the water is 144 cm, the rate at which the height is changing is $-0{\cdot}3$ cm/hr.

By solving the differential equation in part (*a*), show that $h = \left(12 - \frac{1}{80}t\right)^2$. **4**

(*c*) How many days will it take for the tank to empty? **2**

(*d*) Given that the tank has radius 20 cm, find the rate at which the water was being delivered to the vegetation (in cm^3/hr) at the end of the fourth day. **3**

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